

# LACK OF DISJOINTNESS IN GENUS-1 SURFACES: THE PUNCTURED BALLOON THEOREM

Arturo Tozzi

Center for Nonlinear Science, Department of Physics, University of North Texas, Denton, Texas, USA  
tozziarturo@libero.it  
arturo.tozzi@alsnapoli2nord.it

*Take a balloon, that is a genus-one manifold. If you break the jointness by piercing its surface, the hole gets lost and the punctured balloon becomes a genus-0 manifold. Starting from this trivial claim, we prove a topological theorem which plainly states that “the ends of a donut can meet, whilst the ends of a kidney pie cannot”. In this succinct note, we discuss the theorem and its implications in disparate topics such as topological connectedness, gauge theories and the physics of the black holes.*

Two sets are disjoint if they have no elements in common, or in other words, if their intersection is an empty set (Cormen et al., 2001). Many topological manifolds contain disjoint subsets such that  $a \cap b = \emptyset$ . To provide an example, the opposite antipodal points on the genus-0 n-sphere described by the Borsuk–Ulam theorem cannot meet, therefore are disjoint (Matoušek 2003; Peters 2016).

Here we provide a novel theorem, colloquially labelled the punctured balloon theorem (PBT), confronting jointness and disjointness in different genus manifolds.

## Theorem

- An orientable genus-1 surface cannot encompass disjoint points. See the **upper Figure** for a pictorial rendering.

## Proof

- Let's assume that a genus-1 surface contains two (or more) disjoint points, such that  $a \cap b = \emptyset$  (**lower Figures**).
- When a deformation retraction is performed, it unavoidably happens that two (or more) disjoint points a and b (**lower left Figure**):
  - a) Approach the hole border.
  - b) End up closer.
- Because the two (or more) disjoint points a and b cannot meet, they leave an empty region connecting the internal of the hole with the external of the genus-1 surface.
- This would mean that the hole gets lost, being the manifold of genus-0 instead of the genus-1 advocated in the assumption (**lower left Figure**).
- This is a contradiction, hence the assumption must be incorrect.

Therefore, a manifold including two (or more) disjoint points turns out to be a genus-0 manifold. Note that we did not use the term “connected” in our definition of surface, since PBT casts doubts on the role of topological connectedness in the assessment of both sets and subsets. Even though deformation retraction stands for a mapping that captures the idea of continuously shrinking a space into a subspace, PBT suggests that a genus-0 manifold must necessarily encompass (at least two) non-intersecting elements.

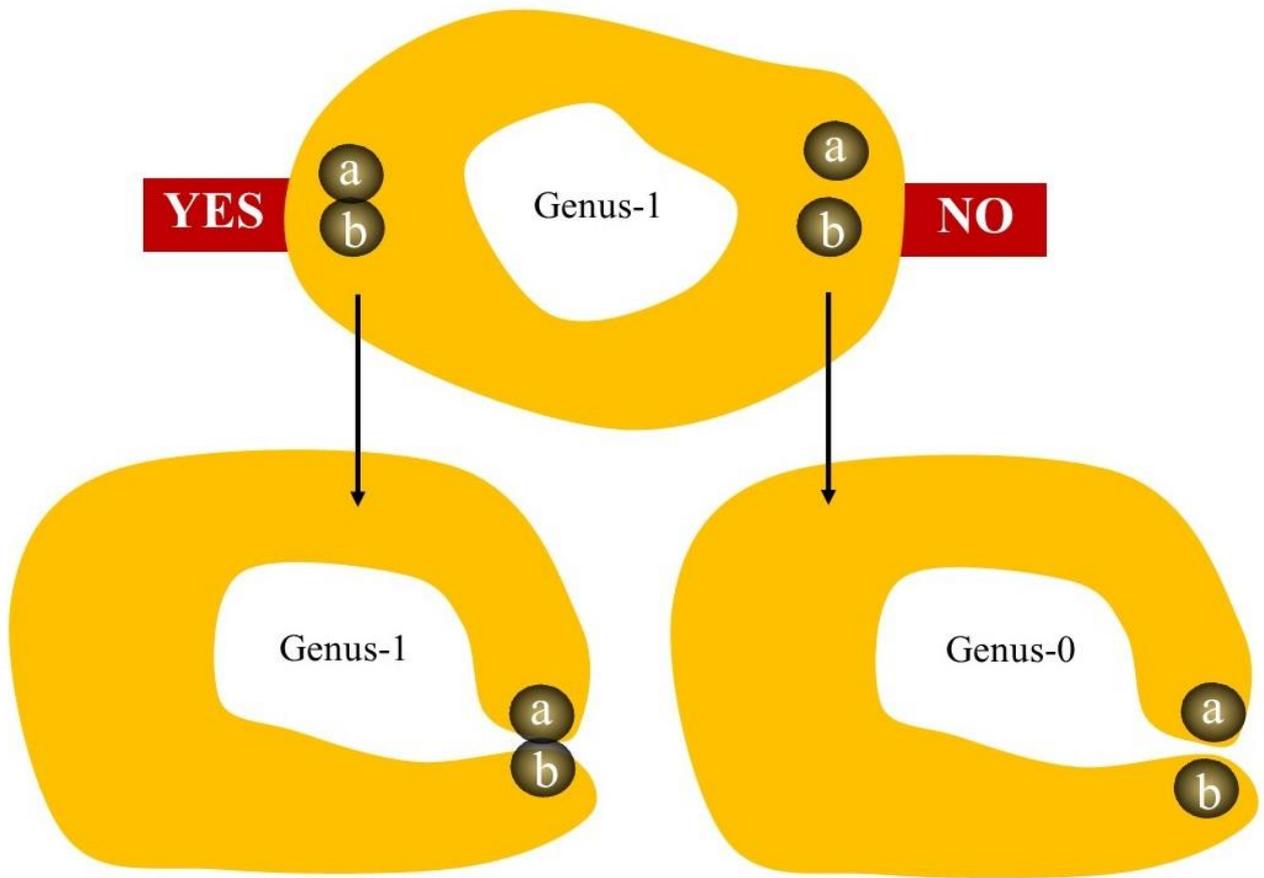
PBT rules out the occurrence of continuity/indivisibility among the elements of a physical genus-0 manifold, holding instead to a discrete operational approach. In touch with this observation, when a choice is performed, e.g., a gauge is introduced in a symmetric manifold (Sengupta et al., 2016), disjoint subsets are generated and manifold connectedness gets lost. This means that a genus-0 manifold, although theoretically shrinkable, must contain some disconnected elements. Take the Poincaré–Brouwer theorem, also termed the “hairy ball” theorem (HBT), stating that there is no non-vanishing continuous tangent vector field on even-dimensional  $n$ -spheres (Eisenberg and Guy, 1979). This implies that the poles of a genus-0 sphere are disjoint from the neighbouring elements. PBT suggests that the vanishing tangent vector fields disappear when the ball is drilled completely through: in this case, since the poles have been removed, we achieve a genus-one torus instead of a ball and HBT does not hold anymore.

PBT implies that a genus-1 surface contains just intersecting, neighborhood points. When shrinking a genus-1 space into a subspace, PBT entails that all the points gather, apart from the ones inside the hole. Therefore, a genus-1 manifold, despite being a set that is not simply connected, must encompass just connected elements, except for the region occupied by the hole.

PBT affects not only disjoint points, but also disjoint shapes, functions, vectors, energies, and so on (for a survey, see Tozzi et al., 2017). In physical terms, PBT suggests that a genus-1 manifold must contain homogeneous elements/features/functions all around the hole, since every part of a genus-1 system is required to topologically intersect with every other part. Therefore, in a physical system encompassing holes - i.e., vortices/antivortices (Padavić et al., 2020), Betti number  $\beta_1 \geq 1$  (Don et al., 2020), topological defects in nanoscopic materials, fragile topology features (Po et al., 2018) and so on - all the regions surrounding the impurities must display at least a few homogeneous and ergodic elements/features/functions.

This also means that a black hole (Chesler 2019) cannot contain a singularity, unless the black hole itself displays genus  $\geq 1$ . If a black hole was a genus-0 manifold (therefore shrinkable to singularity by the huge amount of mass/energy), its own elements would be intersecting. But it runs counter PBT. Hence, the options for a black hole are two:

- 1) Either the black hole is genus-0 and singularities are not allowed.
- 2) Or the black hole is genus  $\geq 1$  and its elements can be compressed down to a tiny region surrounding the hole(s).



**Figure.** A genus-1 surface cannot contain disjoint points (**upper figure**). If two points meet, the hole and the genus are preserved (**lower left figure**). In turn, if two points do not meet, the hole gets lost and the surface turns out to be a genus-0 manifold (**lower right figure**).

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