EEG dynamics on hyperbolic manifolds

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A B S T R A C T

Biological activities, including cellular metabolic pathways, protein folding and brain function, can be described in terms of curved trajectories in hyperbolic spaces which are constrained by energetic requirements. Here, starting from theorems recently-developed by a deceased Field Medal young mathematician, we show how it is feasible to find and quantify the shortest, energy-sparing functional trajectories taking place in nervous systems’ concave phase spaces extracted from real EEG traces. This allows neuroscientists to focus their studies on the few, most prominent functional EEG’s paths and loops able to explain, elucidate and experimentally assess the rather elusive mental activity.

1. Introduction

The scientific knowledge of physical dynamics is currently more advanced than their biological counterparts, the latter still lacking the required explanatory depth. One of the main reasons for this disparity is that, while physics has been provided with powerful mathematical tools, biology has not. Physics works much better in the description of natural features, because we have recognized its subtending manifolds. To make an example, both general relativity and quantum mechanics describe Nature using mathematical structures, such as tensors and probability theory ([1–3]). Their subtending manifolds (i.e., the phase spaces in which the corresponding activities take place) are well-known and have been experimentally confirmed: general relativity is described on a 3 + 1 dimensional pseudo-Riemannian manifold with tensor fields obeying certain partial differential equations (see, e.g., [4,5]), while quantum field theory is portrayed on an R̅ projective Hilbert space, with operator-valued fields obeying certain Lorentz-invariant partial differential equations and commutation relationships [6].

Concerning biology, to make an example, it is still difficult to understand the information content endowed in brain electroencephalogram (EEG) traces, or to find a definition for life [7,8]. There is no unequivocal definition of such biological dynamics, despite many proposals have been suggested. Apart from the intrinsic problems in describing such elusive phenomena, the very current limit is that the most of the proposed features - from the concept of homeostasis to organization, from adaptation to response to stimuli - are just descriptive. In sum, in neuroscience, as in biology in general, we lack the proper manifolds, and therefore we lack mathematically precise objectivity. This is a huge restraint, because the sole language able to describe in quantitative terms scientific issues is the mathematical one. If we leave apart math, we do not have the proper observables, and neural dynamics are made by observables.

The aim of this paper is to make an effort to provide a mathematical, operational, quantifiable definition of the dynamics of EEG real traces. In particular, to assess neurophysiological issues, we need at first to find the proper manifolds where their operations might take place. Therefore, our goal is to describe these phase spaces, by treating brain EEG activities and their corresponding gradient-descent Langevin equation, in terms of algebraic topology.

2. Towards the manifolds of brain activity

In terms of differential geometry treatments of probability measures, the manifold traced out by sufficient statistics displays negative curvature, i.e., it is concave [9,10]. The term manifold refers to a topological space that is locally Euclidean, i.e., a space described by the axioms and postulates of Euclidean geometry, including physical geometry [11]. Moving along this hyperbolic manifold’s phase space (i.e., a space with a geometry of constant negative scalar curvature, in which there are infinitely many lines parallel to a given line [12]) requires a
distance measure that can be granted, for example, by Fisher information metric (Fig. 1). A pair of hyperbolic planes are metrically equivalent, provided they have the same curvature. Is such a hyperbolic geodetic strategy able to describe living (and in particular nervous) dynamics, or is it just an axiomatically-adopted statistical method? In other words, are hyperbolic spaces viable, when we assess brain activity? The answer is affirmative, because the mathematical concept of concave manifold allows the generalization of the classical notion of Euclidean space. Because complex computational models are required in order to capture the cross-scale dynamics typical of neuroscientific issues [13], hyperbolic structures spread the methods of vector algebra and calculus from two- or three- dimensional Euclidean manifolds to functional spaces with every number of dimensions. Also, hyperbolic manifolds such as, i.e., the Hilbert ones, are vector spaces equipped with an inner product that allows the quantification of lengths and angles. The concave spaces are essential tools in far-flung disciplines, from partial differential equations to quantum mechanics, from Fourier analysis (including heat transfer and EEG signal processing) to ergodic theory (including thermodynamics and information entropies). One might argue that the real dynamics of neural activities do not take place on hyperbolic phase spaces, rather on flat ones, such as the Euclidean world. This objection fades away, if we take into account that all the types of dynamics, both physical and biological, can be abstractly described on hyperbolic manifolds. Indeed, according to recent formulations of the Borsuk-Ulam theorem, matching descriptions between biological features can be provided, independent of the curvature of the subtending manifold [14]. In an operational language, this means that the steps of the nervous system’s Langevin equation [15] experimentally detected in the flat, three-dimensional (plus time) Euclidean space, can be transported on a hyperbolic manifold, though, e.g., Levi-Civita or Ehresmann connections [16,17]. Therefore, neural dynamics can be successfully studied on abstract negative-curvature manifolds that mimic the real physical phase space where such processes occur. Here the work of Maryam Mirzakhani, the recently deceased 2014 Field Medal, comes into play, allowing us to calculate the number of trajectories in a hyperbolic phase space.

3. The Mirzakhani theorem

Mirzakhani [18] assessed the “geodesics” of hyperbolic surfaces. On a negative curved surface, geodetics simply stand for the shortest paths between two points. On hyperbolic manifolds, some geodetics are infinitely long (like the straight lines in the Euclidean plane), but others close up smoothly into a loop (like the circumference on a positive-curvature sphere). There are two types of closed geodetics (Fig. 2A and B): the very rare “simple” geodetics, that never intersect themselves, and the much more frequent ones that cut across themselves several times, before closing up. The number of the latter geodetics of a given length L grows exponentially, as their length grows. Mirzakhani demonstrated that also the number of the rare simple geodetics of length L grows, as L gets larger (Fig. 2C and D). However, she showed that, as the length of these simple geodetics grows, their number grows polynomially, instead of exponentially (Fig. 2E). The difference is subtle, but foremost. During an exponential growth, time (or space, or, in our case, the geodetics length) grows as a function $K^n$, where $n$ is proportional to the size of the input and $K$ is a constant [19]. In turn, during a
polynomial growth, time (or space, or, in our case, the geodesics length) grows as a function $n^k$. In the sequel, we will provide the EEG counterparts of the Mirzakhani’s findings, in order to operationalize her rather abstract theorem in terms of neural dynamics.

4. How to turn an EEG trace into a hyperbolic path

As an illustrative example, we assessed real data extracted from EEG responses to external stimuli. The data used in the present study were part of a larger experiment investigating the influence that gender has on brain responses to emotional stimuli [20]. Stimuli were color slides selected from the International Affective Pictures System [21] according to the valence dimension: emotionally positive (valence ratings from 7.2 to 8.2), neutral (valence ratings from 4.4 to 6.2), and negative (valence ratings from 1.3 to 2.0). The analyzed EEG data trace corresponds to a neutral stimulus (picture No. 5731; see Fig. 3A). EEG was recorded using a Quick-Cap with sintered (Silver/Silver Chloride; 8 mm diameter) electrodes. Using the Ten-twenty Electrode Placement System of the International Federation, the EEG activity was monitored over nineteen scalp locations (Fp1, Fp2, F3, F4, F7, F8, T3,T4, T5, T6, C3, C4, P3, P4, O1, O2, Fz, Cz and Pz). All leads were referenced to linked mastoids (A1 and A2), and a ground electrode was applied to the forehead. Additionally, vertical eye movements were recorded with electrodes placed above and below the left eye. Electrode impedance was maintained below 5 mk. The digital EEG data acquisition and analysis system (SynAmps) had a bandpass of 0.15–100.0 Hz. At cutoff frequencies, the voltage gain was approximately –6 dB. The 19 EEG traces were digitized online at 1000 Hz with a gain of 1000 (resolution of 0.84 mV/bit in a 16 bit A to D conversion), and stored on a hard disk. The EEG study was done according with Declaration of Helsinki and was approved by the Ethics Committee of the University of Maribor, Slovenia.

The question is whether the 2D traces detected by the above-mentioned EEG procedure can be detected on manifolds equipped with negative curvature. In other words, is it possible to transport EEG traces onto Riemannian hyperbolic manifolds? The answer is positive, if we perform a parallel transport of the two-dimensional trace on a three-dimensional hyperbolic manifold (Fig. 3B). Although parallel-transport
Fig. 3. How to transfer an EEG trace to a hyperbolic manifold. **Fig. 3A** depicts an EEG trace during and after a neutral visual stimulus. **Fig. 3B**: in this simulation, the two seconds of the EEG traces following the one-second stimulus are transported to a Riemannian, negative-curvature manifold. Therefore, **Fig. 3B** shows what the EEG trace looks like on a hyperbolic manifold. **Fig. 3C-D**: the paths described in **Fig. 3B** are converted in closed paths taking place on hyperbolic manifolds. Note that the path is closed when the EEG trace touches two times the subtending continuous line. **Fig. 3C** depicts an EEG pattern that leaves the central value and spends a long time in intersecting circles, before coming back to same value. **Fig. 3D** depicts an EEG pattern that leaves the central value and spends a short time in a single circle, before coming back to the same value.
requires the solution of a second-order differential equation, analysis shows us that we are allowed to use the first order approximation of the parallel transport, that we pursue in terms of solving geodesic equations for the sufficient statistics [22]. There are different routes one can take to achieve the goal: one needs to resort to one of the following generic transport procedures.

1) Ehresmann connection [17].
2) Levi-Civita connection [23].
3) We can formulate the Hessian operator on the Riemannian manifold in terms of the Laplace-Beltrami operator [24].
4) We can retain a first-order approximation and formulate descent directions that are orthogonal to the previous descent ones, through numerical analysis with the conjugate gradient-descent algorithm [25]. Routinely used in optimization, conjugate gradient descent methods have been used for gradient descent on manifolds traced out by energy functions such as the variational free-energy [26].

Starting from every one of the above-mentioned transport technique, we are able to achieve the mapping of the EEG trace illustrated in Fig. 3A to the hyperbolic Riemannian manifold depicted in Fig. 3B. Our next goal is to describe such EEG hyperbolic paths in terms of the Mirzakhani theorem. In a previous paragraph, we stated that closed surfaces can be detected on a negative curved surface. Therefore, by using the same transport techniques described above, we achieve EEG paths that come back to the same EEG line’s value (Fig. 3C and D). In case of our EEG traces’ paths, such closed surfaces are circles on a hyperbolic manifold: every simple circle (corresponding to a Mirzakhani’s “simple” geodetics) stands for a EEG path that comes back to the same value in less than 20 ms. More complex circles (corresponding to a Mirzakhani’s more frequent geodetics) return to the same point (the central line of the EEG) in a variable, longer time: this means that some EEG traces, before coming back to their central values, describe one or more intersecting circles on the hyperbolic manifold. In other words, EEG traces spend a variable time up or down the central value, and then return to it, describing a variable number of circles which may or may not overlap.

4.1. The math subtending EEG’s neural dynamics

The theorem by Mirzakhani allows us to state that, if the size of a negative curved manifold is progressively increased, curved geodetics of increased length are achieved on its surface. This leads also to a quantifiable increase in their number. In neural terms, the increase in size of the manifold may stand for rather different systems’ dynamical features: e.g., either a growth in energy, or in signal complexity, or in the neural units recruited in the brain during a mental task. Note that the Mirzakhani’s very rare “simple” geodetics are of foremost importance, when translated in a neural context: indeed, they stand for the shortest paths, i.e., the steepest descent trajectory able to attain the system’s nearest free energy minimum. To make a biological example, such kind of short and very efficient energetic paths are the ones that make the folding proteins less frustrated, therefore more efficient in performing their biochemical operations [27]. The Mirzakhani’s framework tells us that, when the available free energy increases in the phase space, the most frustrated, and therefore least-performing, proteins display an exponential growth in their number of folding paths, while the less frustrated a polynomial one. The same holds for experimentally detected EEG traces.

Taking into account that the neural dynamics occur on hyperbolic phase spaces, two different EEG nervous activities can be assessed and carefully quantified: the first one is characterized by countless trajectories that slowly close up into loops, while the second one by less frequent trajectories that more quickly close up into loops. This faster activity, which stands for the best available trajectories from higher to lower free-energy states (according to the thermodynamic dictates), might be very important in order to explain the peculiar dynamics of the human brain’s electric activity detected through EEG techniques. In turn, the slower activity might stand for the noisier, less efficient and more random dynamics taking place in the brain. The Mirzakhani’s approach, when transferred to the neuroscientific realm, provides the recognition and quantification of both the slower and faster nervous activities, allowing researchers to focus efforts and experimental apparatus just on their favored ones. Indeed, these mathematical devices make it possible achieve graphs that illustrate the behavior of both the geodetics occurring in the biological system under evaluation (Fig. 2E).

4.2. Neuroscientific significance

Here we provide a few examples of the effectiveness of such kinds of purely mathematical approaches in the evaluation of biological issues. During a given brain activity (say during perception), several neural assemblies responsible for operations involved in this process are recruited and start to fire [28]. Furthermore, in this process, interactions occur among the members of the assembly (the neurons) as well as between several assemblies in order to mentally present the complex stimulus [29], giving rise to electric paths that take place in a functional hyperbolic phase space. For technical readers, such space can be built and quantitatively assessed through the authoritative tools of differential geometry, parallel transport and dynamics on manifolds described by Sengupta et al. [10]. Some of these trajectories are closed, others are not. In both real and artificial neural networks, these functional loops may follow linear, sigmoid, exponential or logarithmic responses [27,30]. While the most of the closed trajectories follow random walks, a few obey to the dictates of energy optimization, taking the shortest available paths [31]. Despite the random walks of neural ensembles have been proven to be able to encode input stimulus without any specific training [32], nevertheless, the faster, nonrandom, more efficient, polynomial paths of activation described by us are of foremost importance in the brain, because they allow better responses and energy sparing. It is not easy to detect such rare and peculiar polynomial activities among the others. When an increase of the given brain activity takes place (e.g., an increase in perceptive task load) and more neural assemblies are recruited, the closed trajectories on the negative curved manifold grow in number with exponential or polynomial behavior. The feasible operational procedure that we used is the following, provided just as one of the countless examples and possibilities. During a visual perceptual task, brain activity is detected through EEG techniques. At first, we will notice the activation of the visual primary sensory areas, then of the ventral/dorsal visual pathways and frontal areas. The next step is to project such three-dimensional EEG dynamics on a hyperbolic manifold, looking for the closed loops of this brain activity. Our experimental prevision is that, when more neuronal areas will be progressively recruited, we will observe a generalized exponential increase in EEG traces’ loops. Furthermore, we will detect the simultaneous presence of a few loops displaying polynomial growth. The latter stand for the most efficient and powerful EEG activities related to the visual task under investigation. Therefore, due to the Mirzakhani’s results, it becomes easier for neuroresearchers to detect and separate the two different types of EEG closed trajectories: the less useful grow exponentially, the more efficient ones grow polynomially. This allows scientists, when assessing the “mre magnum” of EEG brain oscillations, to “erase” from their analysis all the useless closed trajectories, and to focus just on the more interesting ones.

5. Conclusions

We made an effort to provide a feasible manifold model, where EEG nervous activities might take place, taking into account that physics is very successful in coping with the complexity of our world, while biology still lacks the required mathematical tools. For this reason, instead of inductively analyze the currently available empirical data,
we proposed to start from a very abstract mathematical framework, in order to formulate top-down previsions that can be experimentally tested.

Based on the recent literature, we introduced a negative curvature, concave, hyperbolic phase space where the functional trajectories of nervous EEG dynamics might take place. Such kind of manifold gave us the possibility to use the theorems developed by Maryam Mirzakhani, the first female mathematician honored with the prestigious Fields Medal Award (corresponding to the Nobel Prize for math), died in 2017 at just 40 years. Her approach allowed us to isolate and rigorously quantify the evolution of two different kinds of EEG neural responses: the exponential and the polynomial ones. It is noteworthy that, during a dynamical process taking place on a negative curvature manifold, at first the increase in polynomial growth is higher, when compared with the exponential one. In turn, when the dynamical process recruits more systems' units, the exponential growth prevails. Our approach suggests that, when assessing the dynamical activity of artificial or real neural networks, it would be preferable to focus the neuroresearch on the more efficient and significant polynomial responses, rather than the linear, sigmoid or exponential ones. Indeed, polynomial functions exhibit many desirable theoretical properties that make them best-suited for the study of nervous activity.

References