THE INFORMATIONAL ENTROPY ENDOWED IN CORTICAL OSCILLATIONS

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ABSTRACT

A two-dimensional shadow may encompass more information than its corresponding three-dimensional object. Indeed, if we rotate the object, we achieve a pool of observed shadows from different angulations, gradients, shapes and variable length contours that make it possible for us to increase our available information. Starting from this simple observation, we show how informational entropies might turn out to be useful in the evaluation of scale-free dynamics in the brain. Indeed, brain activity exhibits a scale-free distribution that leads to the variations in the power law exponent typical of different functional neurophysiological states. Here we show that modifications in scaling slope are associated with variations in Rényi entropy, a generalization of Shannon informational entropy. From a three-dimensional object's perspective, by changing its orientation (standing for the cortical scale-free exponent), we detect different two-dimensional shadows from different perception angles (standing for Rènyi entropy in different brain areas). We show how, starting from known values of Rènyi entropy (easily detectable in brain fMRIs or EEG traces), it is feasible to calculate the scaling slope in a given moment and in a given brain area. Because changes in scale-free cortical dynamics modify brain activity, this issue points towards novel approaches to mind reading and description of the forces required for transcranial stimulation.

Keywords:
Rényi entropy, power laws, nervous system, scale-free, shadows, central nervous system

INTRODUCTION

Counter-intuitively, a 3D object might encompass less information than a 2D one(we do not see the hidden side of the moon). When observed from a given standpoint, an opaque 3D object could be less informative than a series of its 2D shadows (**Figure 1A**). Here we ask: is there a way to increase the information content of the shadow, to make it possible the detection of more details about the object, let us say a toy? The answer is positive, if we rotate the toy clockwise (or anticlockwise) along its central axis (**Figures 1B, 1C**). In this case, we achieve a pool of different shadows and shape contours with various angulations, able to increase our total available information. Therefore, 3D object's shadows (taken in its totality) are more informative, for a watching observer, whenever he moves. This "toy" analogy gives us the possibility to assess brain function in a novel way. For the purpose of this paper, the toy at rest stands for a given brain scale-free exponent, and its shadows for the corresponding Shannon entropy. In turn, the changes in location of the moving toy stand for changes in brain scale-free exponents, while the modifications in shadow's shape and its undulating contour stand for the corresponding Rènyi entropy (Rènyi 1961; Rènyi 1966; Peters 2017; Peters-Ramanna 2016).

Using our example of a toy as a 3D object and its moving shadows, the aim of this paper is to show how it is feasible to extract information about such toy, by looking at its moving shadows: in neurophysiological terms, we want to assess the relationships among scale-free exponents and generalized informational entropies in the brain. Furthermore, we would consider the recent advancement in the neurophenomenology research that established a link between a nested operational architectonics hierarchy of brain functioning (indexed by the EEG field) and the phenomenal (mental) level of brain organization (Fingelkurts and Fingelkurts, 2001; Fingelkurts et al., 2010). According to this line of research, the meanings, which subjectively (mentally) are experienced as thoughts or perceptions, can best be described objectively as created and carried by dynamic fields of neural activity of different sizes that are nested within the operational architectonics hierarchy that organize hundreds of millions of neurons and trillions of synapses (Fingelkurts et al., 2009, 2013a). We perform simulations in order to demonstrate how, starting from the Rènyi entropy easily detectable in neurodata series (e.g., from brain fMRI or EEG), it is feasible to calculate the cortical scale-free exponent in a given moment, and, consequently, the corresponding mental state.

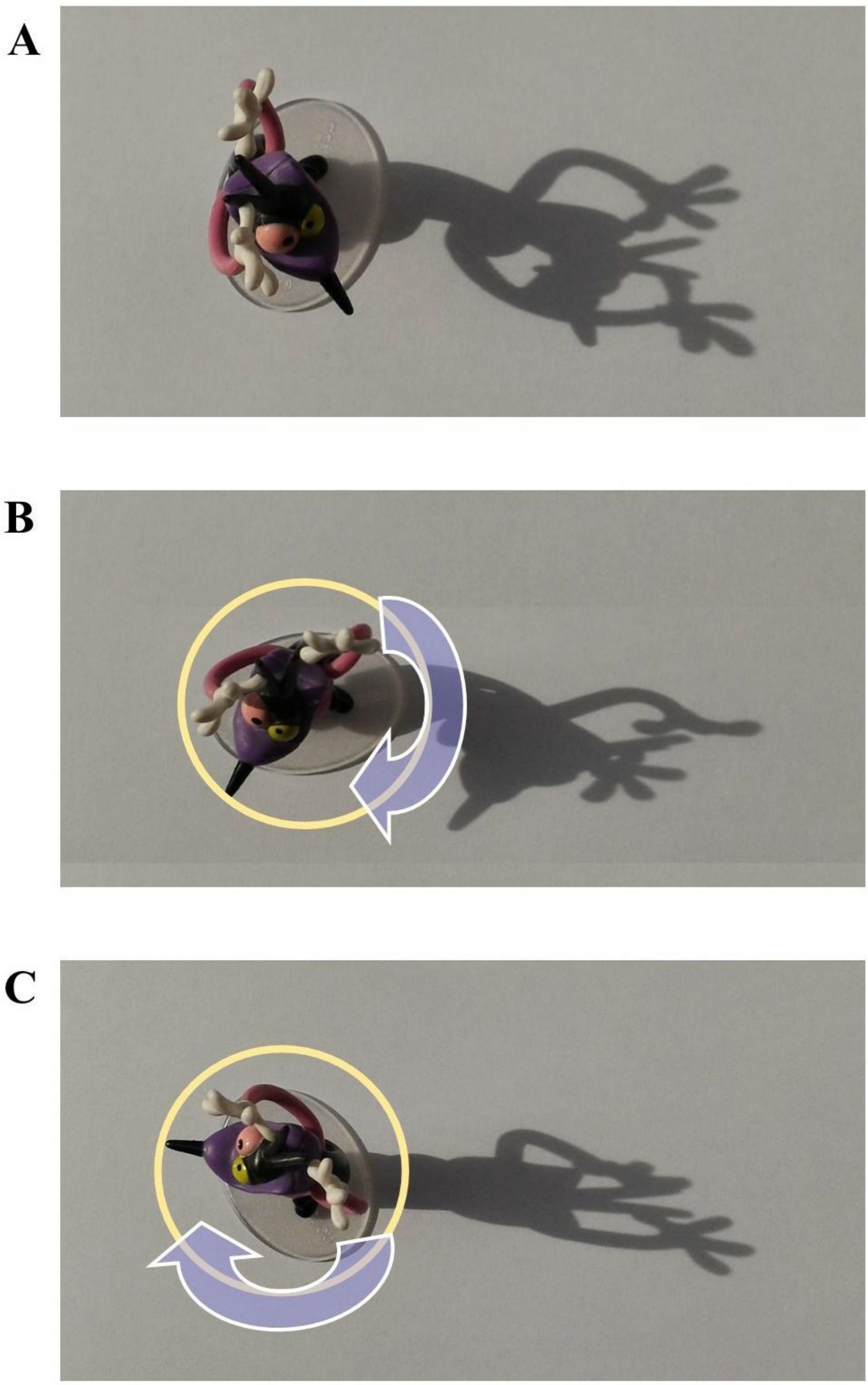

Figure 1. A toy under a light source. In **Figure 1A**, the 2D shadow offers more details of the 3D object than the view on the toy from above. **Figures 1B** and **1C** show how toy rotations lead to shadows with different shapes, and, consequently, with different information content. Therefore, a complete 360 degrees rotation of the object allows a complete evaluation of the shadows' shapes.

SHANNON ENTROPY ON A TOY

The brain activity, e.g., in our analogy the toy and its movements, can be described in terms of scale-free dynamics. Indeed, the brain activity observed at many spatiotemporal scales exhibits fluctuations with complex scaling behavior (Newman, 2005; Fingelkurts et al., 2010, 2013a), including not just cortical electric oscillations, but also membrane potentials and neurotransmitter release (Linkenkaer-Hansen et al., 2001; Fox and Raichle, 2007; Milstein et al., 2009). In particular, the frequency spectrum of the cerebral electric activity displays a scale-invariant behavior, characterized by a power spectrum, a frequency and an exponent that equals the negative slope (which we will term "scaling slope") of the line in a log power versus log frequency plot (Pritchard, 1992; Van de Ville et al., 2010). In simple terms, a scaling slope 1= stands for a horizontal line parallel to the x axis: the steeper the line, the higher the scale-free behaviour. The (spatial) fractals and (temporal) power laws can be regarded as intrinsic properties of the brain and characterize a large class of neuronal processes (de Arcangelis and Herrmann, 2010; He et al., 2010; Deli et al., 2017); moreover, pink noise distributions contain information about how large-scale physiological and pathological outcomes arise from the interactions of many small-scale processes (Jirsa et al., 2014).

Keeping this in mind, a given slope in a given moment of brain activity stands, in our example, for the toy at rest. Toy features can be described through their shadow, e.g., the Shannon entropy. The classical informational Shannon entropy is (Shannon 1948):

$$\mathrm{H(X)} = -\sum_{x \in X} \Pr(x) \log_2 \left[\Pr(x) \right].$$

The term X is a random variable with *n* possible outcomes, while $p_i = Pr(x)$, for *i*= 1, 2, 3, …*n*, is a probability distribution *Pr* on a finite set. Shannon entropy states that, under ergodic conditions, if we know the values of *Pr(x)*, we may obtain the values of $\mathrm{H(X)}$. In other words, $\mathrm{H(X)}$ is a probability density function that defines a generic probability distribution *Pr*such that, if we modify *Pr*, we obtain a different value of entropy on Shannon's curve. Hence, changing the probabilities usually occurs when we have prior information. In terms of the object depicted in **Figure 1A**, the steady toy stands for the systems' microstates, undetectable by macroscopic observers, while the shadow stands for the lower-dimensional Shannon entropy. Looking at the shadow (the Shannon entropy), we achieve otherwise undetectable information about opaque 3D objects, such as the toy features.

Now we need to find a brain functional counterpart for the moving toy, its changing shape and varying contours of shadows. Indeed, the cortical scale-free slopes are not constant, rather they display multiple possible exponents during various mental states. Despite a few skeptical studies (Tribukait and Eiken, 2016), it is widely accepted that different functional states - spontaneous fluctuations, task-evoked, perceptual and motor activity (Buszaki and Watson, 2012), cognitive demands (Buiatti et al., 2007; Fetterhoff et al., 2014), ageing (Suckling et al., 2008) - account for variations in scale-free exponents across cortical regions (Tinker and Velazquez, 2014; Wink et al., 2008). Accordingly, we may view brain activity as an ensemble of intertwined (mono)fractals, each with its own power slope and scaling range, each marking dynamical transitions between different response regimes (Papo, 2014; see also review Fingelkurts et al., 2013a). In terms of our toy, the whole brain activity stands for all the possible 360 degrees of rotation and their corresponding shadows.

INTRODUCING RENYI ENTROPY

The 360 degrees changes in toy angulation that gives rise to different shadows can be described in terms of Shannon entropy generalizations. Among the available ones (e.g., Tsallis, 1988), here we favour the one-parameter class of Rényi entropies (Rényi, 1966), a flexible and underestimated information-theory based index (Cambell, 1965). Rényi entropy (1961) is characterized by a changing parameter α andcan be defined as:

$$H_\alpha(X) = \frac{1}{1-\alpha} \sum_{\substack{i=1, \\ x \in X}}^{N} p_i^\alpha(x),$$

where $0 \le \alpha < \infty$ and $p_i^\alpha(x)$ is the probability of event x. The Rényi entropy approaches the Shannon entropy as α approaches 1. In other words, the Shannon entropy is a case of Rényi entropy in which α= 1. Rényi entropy has applications in dynamical systems (Hentschel and Proccacia, 1983), coding (Cambell, 1965), information transfer (Jizba et al., 2012), theories in quantum mechanics, black holes' mutual information (Dong, 2016), assessment of human heartbeat fluctuations and coding/noncoding DNA sequences analysis (Costa et al., 2005). Rényi entropy has also been used to quantify ecosystems dynamics and diversity: from land cover types (Carranza et al., 2007; De Luca et al., 2011), to coastal dunes environments (Drius et al., 2013), from urban mosaics (Carranza et al., 2007) to species diversity in large areas (Rocchini et al., 2013).

WHY THE RENYI ENTROPY, INSTEAD OF THE SHANNON'S ONE?

This paragraph will be devoted to elucidate why Rényi entropy should be preferred, in order to assess brain activity. Shannon entropy depicts the discrete probabilities of an event in terms of a single curve. However, in many physical and biological cases, diversity cannot be reduced to a single index information, since all its aspects cannot be captured in a single statistic (Gorelick, 2006).Rényi entropy, in turn, depicts the discrete probabilities of an event in terms of many curves. Therefore, unlike Shannon entropy, Rényi entropy makes it possible to describe the system's status not just at a specific moment, but also when its trend *varies with time* (Müller et al., 2000; Patil and Taillie, 2001). Indeed, a complete characterization of landscape diversity can be achieved if, instead of a single index, a parametric family of indices is used, whose members have varying sensitivities to the presence of rare and abundant elements (Jost, 2007). Unlike Shannon entropy, which is single-valued, Rényi entropy is useful in measuring a range of information levels such as those in A-yeh and Peters (2016), the dynamics of non-stationary processes (Shaymov, Fradov, 2016) and the measuring of the distribution of entropy changes over time (see, *e.g.*, Jauregui, Zunino, Lenzi, Mendes, Ribeiro, 2018). In other words, Rényi entropy offers a "continuum of possible diversity measures" (Ricotta and Avena, 2003) at diverse spatiotemporal scales, becoming increasingly regulated by the commonest when α gets higher. The change in α exponent can be regarded as a scaling operation that takes place not in the real, but in the data space (Podani, 1992): we are thus allowed to evaluate how changes of Rényi parameter influence the structure of information measures in the probability space of the scale-free dynamics. Diversity profiles are flat when the landscape is even, and steeply decrease as the landscape turns uneven (Jost, 2010). Therefore, Rényi's formulation allows a continuum of possible diversity measures which differ in their sensitivity to the rare and abundant elements in a landscape.
Because of its build-in pre-disposition to account for self-similar systems, Rényi entropy is an effective tool to describe multifractal systems (Jizba and Arimitsu, 2001). The generalized fractal dimensional and the Rényi exponent can be thought of as interchangeable. In technical terms, the Rényi parameter is connected via a Legendre transformation with the multifractal singularity spectrum. See Jizba and Arimitsu (2001) and Jizba and Korbel (2014) for technical details on mathematical proof. In particular, this means that Rényi entropy and scale-free exponent (and therefore the scaling slope) are strictly correlated. Therefore, changes in scaling slope lead to changes in the Rényi parameter, and vice versa (Słomczynski et al., 2000): scale-free systems lead to different probability outcomes, based solely on increases or decreases of the scaling slope. In sum, the "probabilistic" virtues of Rényi entropy represent a novel physics-based approach to probe brain scale-free dynamics, with the potential to lead to new insights into brain systems at all space-time scales and all levels of complexity.
Once elucidated why Rényi entropy is more suitable than Shannon's, in the next paragraph we will try to answer to the crucial question: what for? How might such a scheme help, in the experimental assessment of brain pink noise? Can the variations (fluctuations) in Rényi entropy can be correlated with the values of a circle perimeter (standing for 360 degrees angles of our toy example rotations)? Our moving toy might help to elucidate the issue. Indeed, toy rotations form an angle on a 360 degrees circumference, so that every angle corresponds to a different shadow, i.e., to a different value of Rényi entropy (**Figures1B** and **1C**). In neurophysiological terms, we are allowed to calculate the changes in brain scaling slopes, just by evaluating a full spectrum of the corresponding Rènyi entropies. The procedure will be described in the next paragraph.

WORKING ON RANDOMLY GENERATED ANGLES

Here we show the procedure in order to correlate our formulation of Rényi entropy with brain scale-free dynamics. We performed simulations of randomly generated angles (Çankaya et al., 2015) on a 2D circle. Every angle displays a random degree, so that their total sum stands for a 360 degrees full circle. The number of rays originating from the circle's center can be modelled through the cosine theorem, which states that:

$per_i^2 = b_i^2 + c_i^2 - 2b_ic_iCos(\theta), i = 1,2, \ldots n,$

where b and c (which could be randomly generated in simulations) are the length of rays originating from the circle's central point. The parameter per is one of the three sides of the triangle lying on the circle perimeter (**Figure 2, left side**). The randomly generated parameter θ stands for the angle between two rays, while n stands for the number of rays. The sum of per_i, namely $\sum_{i=1}^{n} per_i$, can be regarded as the circle's perimeter. Each set of perimeter values produces just one R**è**nyi entropy value: therefore, we replicated the simulation one hundred times.

Note that the fluctuations in Rényi entropy values depend just on the different perimeter values. In terms of brain assessment, we evaluate the cortical scale-free activity and the scaling slopes instead of evaluating random numbers. In such a vein, two cerebral hemispheres can be unfolded and flattened into a two-dimensional reconstruction by computerized procedures (Van Essen, 2005). In this model, the nervous electric activity occurs on a 2D brain surface with an approximated circular shape (**Figure 2, left side**). Perimeter values allow us to detect the nervous activity's travel lengths on the brain surface. After the brain signal's travel lengths have been achieved, the corresponding Rènyi Entropy values can be computed ("EntropyEstimation" package: https://cran.r-project.org/web/packages/EntropyEstimation/EntropyEstimation.pdf). **Figure 2, right side,** displays simulated Rényi values obtained from the perimeter values computed through the cosine theorem. In toy's terms, this means that a larger rotation gives rise to higher levels of information detectable from the shadows, while, in brain's terms, the higher the variations in scaling slopes, the higher the detected Rényi entropy and vice versa.

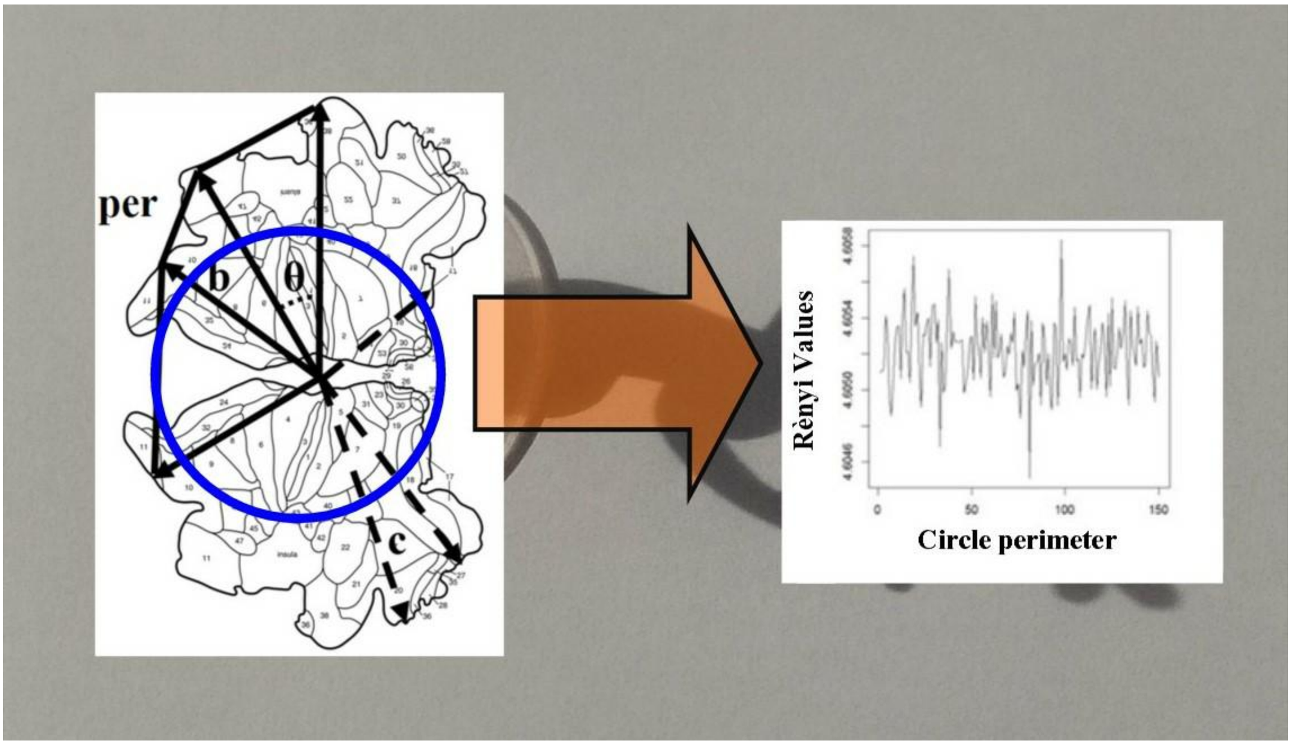

Figure 2. Renyi entropy' circle perimeter and its values (computed by the Renyi.z function in version 3.2.3 of R software) are superimposed, respectively, to the moving toy and its shadow. **Left part**: Cosine theorem for the evaluation of travel length Rényi entropies in a model of two-dimensional brain (modified from Van Essen, 2005). **Right part:** Simulated Rényi values for each circle's perimeter values for n=60 and a Rényi exponent=4; θ is generated from a uniform distribution in (0,1) and multiplied by 360. The Rènyi entropy values' bandwidth is around 4.6.

CONCLUSIONS

We showed that a clear correlation does exist in the brain between cortical scaling slope and brain activity evaluated in terms of probability states. Our approach enables us to achieve unknown scale-free exponents from known entropy changes, allowing us to evaluate the macro-states of the nervous system based on a sole order parameter, although we lack a perfect knowledge of its micro-states. In our simulations, random numbers generated from different distributions are able to modify the values of Rényi entropy: this means that the brain scale-free spectrum can be extrapolated, just starting from the values of entropy experimentally detected in EEG or fMRI traces. Considering the connection between brain activity dynamics and the mental states (Fingelkurts and Fingelkurts, 2001; Fingelkurts et al., 2009, 2010, 2013a), this observation paves the way to tackle the difficult issue of mind reading.

We hypothesize that, in order to optimize its functions, the brain could be equipped with intrinsic mechanisms of scaling features: in this framework, changes in scale-free exponents play a crucial role in information processing, leading to variations in entropy and probability of mental states. Empirical evidence suggests that cognitive tasks are modulated by the scale-free exponents of the brain fluctuation probability function, leading to a shrinking of multifractal spectrum and/or transitions from mono- to multi-fractal distributions (Popivanov et al., 2006; Fraiman and Chialvo, 2012).

This approach paves the way for novel approaches to already existing neural network models (Gravier et al., 2016; MegamNgouonkadi et al., 2016) and for innovative diagnostic and therapeutic strategies. For example, the widely-diffused method of pairwise entropy in neuroimaging techniques (Watanabe et al., 2014; Peters et al., 2017) might benefit of theRényi entropy-based approach. It is indeed feasible to start from known parameters of probability distribution, in order to achieve the unknown values of multifractal slopes. In particular, a change in Rényi exponent, caused by variations in scaling slopes, might lead to different probability outcomes, which can be calculated if just their pairwise entropies are known.
We also conjecture that the scale-free-like brain electric activity - setting aside their supposed relationships with self-organized criticality (Bak et al., 1987; Tozzi, 2015), nonequilibrium steady-state dynamics or second order phase transitions (Papo, 2014; Perkins et al., 2014; Tozzi, 2014; Tozzi and Peters, 2016) - could be modulated through the superimposition of an external electric currents characterized by carefully chosen scale-free exponents. This may have a practical application. For example, our results suggest that transcranial electrical stimulation’s techniques need to take into account not only the amplitude and frequency of the applied waveforms (Reato et al., 2013), but also their scaling slope. A further tenable field of application are the diseases that have been linked to disturbance of brain networks - Alzheimer’s disease, depression, attention deficit hyperactivity disorder, schizophrenia, autism (Fox and Raichle, 2007; Fingelkurts and Fingelkurts, 2010) -, meaning that they could be ameliorated, or even treated, by appropriate exogenous electromagnetic fields - e.g., via selective application of electric/magnetic waves of specific scale-free exponent on target micro-areas - able to “recover” and restore the physiological brain functions (Sunderam et al., 2009; Fingelkurts and Fingelkurts, 2010, 2015) and related subjective experiences (Fingelkurts et al., 2013b).

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