1° Give an example of a pair of shapes A, B with either matching or strongly similar descriptions $\Phi(A), \Phi(B)$ so that one shape can deformed into the other shape.
2° Write a Mathematica script to illustrate Part 1.
3° Give a counterexample to disprove Conjecture 5.46 that, for all pairs of shapes A, B with matching or strongly similar descriptions $\Phi(A), \Phi(B)$, one shape can be deformed (mapped into) another shape, provided the shapes that have matching descriptions.
4° Write a Mathematica script to disprove Conjecture 5.46. ■

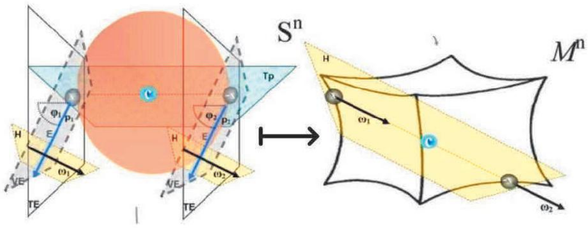

Fig. 5.20. $S^n \mapsto M^n$

5.10 Borsuk-Ulam Theorem Extended to hyperbolic surfaces by A. Tozzi

The Borsuk-Ulam theorem (BUT) tells that there exists a pair of antipodal points on convex n-sphere that are mapped to the same point in n-dimensional Euclidean space $\mathbb{R}^n$. In this section, let S^n be a convex manifold containing a pair of antipodal points that map to a single point in $\mathbb{R}^n$.

So far, the features of geometric regions in 2D and 3D digital images have features such as oblong, concave, diameter, perpendicular, circular that are mapped to real numbers that quantify region features. In this section, we consider image regions equipped with hyperbolic geometry characterized by sectional curvature -1, *i.e.*, regions with a concave shape. The question to consider now is where the antipodal points on flat or on concave surfaces (predicted by BUT) can also be found on image surfaces with negative curvature. In other words, is it possible to transport antipodal points found by BUT to corresponding vectors on a Riemannian manifold M^n with negative curvature?

The answer to this question is positive, provided we map (parallel transport) the antipodal points identified by BUT to points on a hyperbolic manifold. Although parallel transport requires the solution of a second-order differential equation, analysis shows us that we are allowed to have a first-order approximation of the parallel transport. In effect, we pursue parallel transport by solving geodesic equations for sufficient statistics [94] (see, also, an overview of the important aspects of geometric numeric integration by E. Hairer [93]). To solve the parallel transport problem, a number of options are available.

$\boxed{\text{Parallel Transport Options}}$

1^o Ehresmann connection [68].
2^o Levi-Civita connection [20].
3^o We can formulate the Hessian operator on the Riemannian manifold in terms of the Laplace-Beltrami operator [111].
4^o 4) We can also retain a first-order approximation and formulate descent directions that are orthogonal to the previous descent ones, through numerical analysis with the conjugate gradient-descent algorithm [112]. Routinely used in optimization, conjugate gradient descent methods have been utilized for gradient descent on manifolds traced out by energy functions such as the variational free-energy [218]. $\blacksquare$

Example 5.54 walks through the steps for parallel transport using the Ehresmann connection.

Example 5.54. **Antipodal points on S^n mapped to Riemannian manifold M^n.**
Parallel transport of antipodal points (predicted by BUT) on convex S^n to antipodal points on a concave Riemannian manifold M^n is illustrated in Fig. 5.20. This is accomplished with an Ehresmann connection (for the details, see C. Ehresmann [68]. After the Ehresmann connection has been performed and the vectors ω_1, ω_2 are found, then use the Riemannian exponential map $exp(\omega_i), i = 1, 2$ from S^n to M^n and the logarithmic map $log(\omega_i), i = 1, 2$ for the opposite mapping $M^n \mapsto S^n$. Let H in Fig. 5.20 be the plane on the Riemannian manifold M^n, containing the vectors ω_1, ω_2 that represent the antipodal points $x, -x$ on S^n. After this has been achieved, use the ham-sandwich theorem to find the center of the hyperbolic manifold (denoted by $\bigcirc$).

Theorem 5.55. Ulam's Ham Sandwich Theorem.
For any three given sets in Euclidean space, each of finite outer Lebesgue measure, there exists a plane that bisects all three setss, i.e., *separates each of the given sets into two sets of equal measure.*

Proof. Assume that the Borsuk-Ulam Theorem holds and then the proof of the ham-sandwish follows (for the details, see [15]). $\square$

$\blacksquare$

In summary, the Borsuk-Ulam Theorem can be extended to concave surfaces, making it possible to look for antipodal points in images equipped with a negative curvature.