THE PHYSICAL MEANING OF J-FUNCTION ELUCIDATES THE MONSTROUS MOONSHINE CONJECTURE

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ABSTRACT
Monstrous Moonshine refers to the unexpected correlation between two apparently incommensurable mathematical entities, *e.g.*, the j-function and the dimensions of the Monster Module. In order to elucidate the relationships between modular meroporphic functions and group theory, we embedded physical coordinates into the upper half plane of the complex numbers, *e.g.*, the Argand diagram of the j-function and Riemannian surfaces of genus O. We achieved values compatible with the hypothesis of a flat Universe subtended by the Monster Module. We provide evidence that the Monstrous Moonshine conjecture might display a quantifiable physical counterpart, *e.g.*, the spatial curvature index, and discuss the implications in cosmology.

The Fischer-Griess Monster Group M of order 10^{54}, the largest of the 26 sporadic groups of the Conway's atlas of finite simple groups (Conway and Norton, 1979), is equipped with a large number of dimensions. Indeed, 196,883 is the smallest-dimensional space in which this object can be represented (du Sautoy, 2008). Every group is a nonempty set with a binary operation $\circ$ defined on it. With no special properties of $\circ$, the pair (M, $\circ$) is called a monster groupoid. To achieve a Monster group, we assume that $\circ$ is associative, M has an identity element and that every element of M has an inverse. Frenkel, Lepowsky and Meurman showed that the automorphism group of the monster module (as a vertex algebra) is M (Frenkel et al., 1984).
The term *Monstrous Moonshine* describes the unexpected connection between some elements of M and special modular functions, in particular the Klein j-invariant function. The modular function j is the generator or hauptmodul of the genus zero function field. Originally, the j-function was defined by (Lehmer, 1942) by

$j(\tau) = 1728J(\tau)$, whose Fourier development is

$j(\tau) = e^{-2\pi ir} + 744 + 196884e^{2\pi ir} + 21493760e^{4\pi ir} + \cdots$, where

$\mathrm{I}[\tau] > 0$ is the half-period ratio and

$$\mathrm{J}(\tau) = \frac{4}{27}\frac{\left[1-\lambda(\tau)+\lambda^2(\tau)\right]^3}{\lambda^2(\tau)\left[1-\lambda(\tau)\right]^2}$$

is Klein's absolute invariant and $\lambda(\tau)$ is the eliptic lambda function (Piezas)

$$\lambda(\tau) \equiv \frac{\vartheta_2^4\left(e^{\pi ir}\right)}{\vartheta_3^4\left(e^{\pi ir}\right)}, \text{ where}$$

$\vartheta_i\left(0,q\right)$ are Jacobi theta functions and

$q = e^{\pi ir}$ is the nome and $1728 = 12^3$.

Indeed, 196,884 is the first coefficient in the Fourier expansion of the normalized elliptic modular invariant (Gannon, 2006). In sum, the Monstrous Moonshine conjecture suggests a puzzling relationship between one of the first terms in the Fourier expansion of j(τ), e.g., the hauptmodul J, which value is 19884, and the simple sums of dimensions of irreducible representation of the Monster group M, which is 196883 (Conway and Norton, 1979).

MONSTER'S PHYSICAL COUNTERPARTS

In the effort to elucidate the unusual relationship described by the Monster Moonshine, we took into account the proposed link between the Monster and physics. Indeed, despite the fact that the Monster Module plays no role in the Standard Model, it has been suggested that it might be built into our Universe at a fundamental level (Gannon 2006; Duncan et al., 2015). Moonshine can be regarded as a collection of related examples where algebraic structures have been associated with automorphic functions or forms. The conjecture displays relationships both with the Lie group E8(C), and a lattice vertex operator algebra equipped with a rank 24 Leech lattice (Borcherds, 1992). In such a vein, links have been suggested between Monster features and string theories. In Frenkel-Lepowsky-Meurman construction (Frenkel et al, 1988), the Monster stands for the symmetry of a string theory for a Z^2-orbifold of free bosons compactified on a 24-dimensional torus R^{24}/Λ (Λ denotes the Leech lattice), in the framework of a conformal field theory (CFT) characterized by partition function corresponding to j (Di Francesco, et al., 1997). Furthermore, Witten proposed that pure quantum gravity in AdS_3 (anti De Sitter) space with maximally negative cosmological constant is dual to a holomorphic CFT, where the partition function j-744 of the Moonshine comes into play (Witten, 2007). There also exists a conjectural connection between gravity and Moonshine, via Rademacher sums (Duncan and Frenkel, 2011). Witten also suggests a link, via Virasoro primary fields, between the Bekenstein-Hawking semiclassical entropy estimate for black holes mass and the Monster Module. Last, but not the least, has been proposed that Moonshine is not limited to M, but also to other groups, e.g., the Mathieu group correlated with the umbral Moonshine (Eguchi et al, 2011).
Taking into account these observations, we looked for physical counterparts to plot onto the upper part of the Argand diagram for complex numbers, where modular functions lie. By correlating drawings of the j-function with the Monster Module dimensions and cosmological parameters, we demonstrate that the Monster Module and the meromorphic functions are two sides of the same coin, *i.e.*, both subtend our Universe.

ENCOMPASSING THE MONSTER ONTO J-FUNCTION PLOTS

This Section illustrates how to embed the j-function into a 3D plot which encompasses the 196,844 dimensions standing for the Monster Module. The first step is to build, for different values of j-functions, 2D and 3D graphs of complex numbers (**Figure 1**). The x-axis is considered for a small range of the reals, namely, x in [-744, 744]. The crucial zone to investigate on the x-axis is between-0,5 and +0,5: this window is the fundamental domain of the modular group acting on the upper half plane. We then changed the parameters displayed by the y and z axes (**Figure 2**). In this novel framework, the z-axis, where the values of j-function lie, stands for an infinite number of spatial dimensions. Once achieved the point on the z-axis corresponding to the required dimension of 196,884, we attain the z-plane corresponding to the Monster Module dimensions and its vertex operator Virasoro algebra. The Monster Group, e.g., the automorphic Lie group acting on the Monster Module, performs its rotations into such z-plane. This means that the Monster group encompasses all the events which occur just on the z-plane corresponding to the Monster Module's dimensions. When

we expunge all the z-dimensions apart from the required 196,884 dimensions z-plane, we accomplish the 2D areas in which the Monster Module and its actions take place. In other words, we embedded the Monster in a phase space of the j-invariant. It means that, in an infinite-dimensional space, the trajectories elicited by j- function are correlated with the Monster Module and its dimensions.

The y-axis, corresponding to the value 196,884 $J(\tau)$ in the j-function plot, stands for the values of the spatial curvature index (**Figure 2**), which is the constant representing the curvature of the space in Friedmann-Robertson-Walker metric (Veneziano, 1998). The 2D and 3D plots of j-function show that values higher than 0 on the y-axis give rise to positive curvatures, while values lower than 0 give rise to negative ones (**Figure 3**). The Monster Module is located in a zone of the graph corresponding to a spatial curvature index close to zero. This is in touch with experimental cosmological observations, which describe a flat Universe. The Monster Module plot displays also a homogeneous and isotropic pattern around the values of the cosmological constant = 0, a pattern which disappears when approaching 1 and -1 values. This is in touch with experimental data from the cosmic microwave background, which describe our Universe as ergodic, homogeneous and isotropic (Gawiser and Silk, 2000; Fixsen, 2009). In other words, based on these empirically testable theoretical development, it is feasible that the Monster Module underlies our Universe.

Our scheme takes also into account the Witten's conjecture of a duality between pure quantum gravity and external holomorphic conformal Field theory. Witten suggests the existence of a hyperbolic anti-De Sitter Universe equipped with a strongly negative cosmologic constant. Although it is not the case of our current flat Universe, where the cosmological constant is close to zero, it has been proposed that the spatial curvature index was maximally negative at time 10^{-33} seconds after the Big Bang, thus allowing an earlier anti-De Sitter Universe. In such a framework, our Moonshine conjecture is still valid, provided we slightly modify our y-axis scale. In order to evaluate a possible anti-De Sitter Universe, which requires strongly negative values of spatial curvature index, it is possible to trace logaritmic values on the y-axis.

CONCLUSIONS

We showed how the j-function and the Monster Module, e.g., the Monster vertex operator Virasoro algebra, can be embedded in the same plot. Encompassing the Monster parameters in a j-function plot allows us to provide a physical bridge between the apparently incommensurable Monster Module and meromorphic functions. We showed how, in an infinite-dimensional space, the action of the j-function is correlated with the Monster Module, equipped with 196,884 dimensions. Looking more closely at the domain and range of the j-function, it would seem that there is a relationship between meromorphic plots, Monster Module's dimensions and some features of the Universe, *e.g.*, its curvature and isotropy. In such a vein, the Universe could be equipped with a Monster Module where the actions are guaranteed by the Monster group, e.g., the automorphism group of a vertex operator algebra (Gannon).

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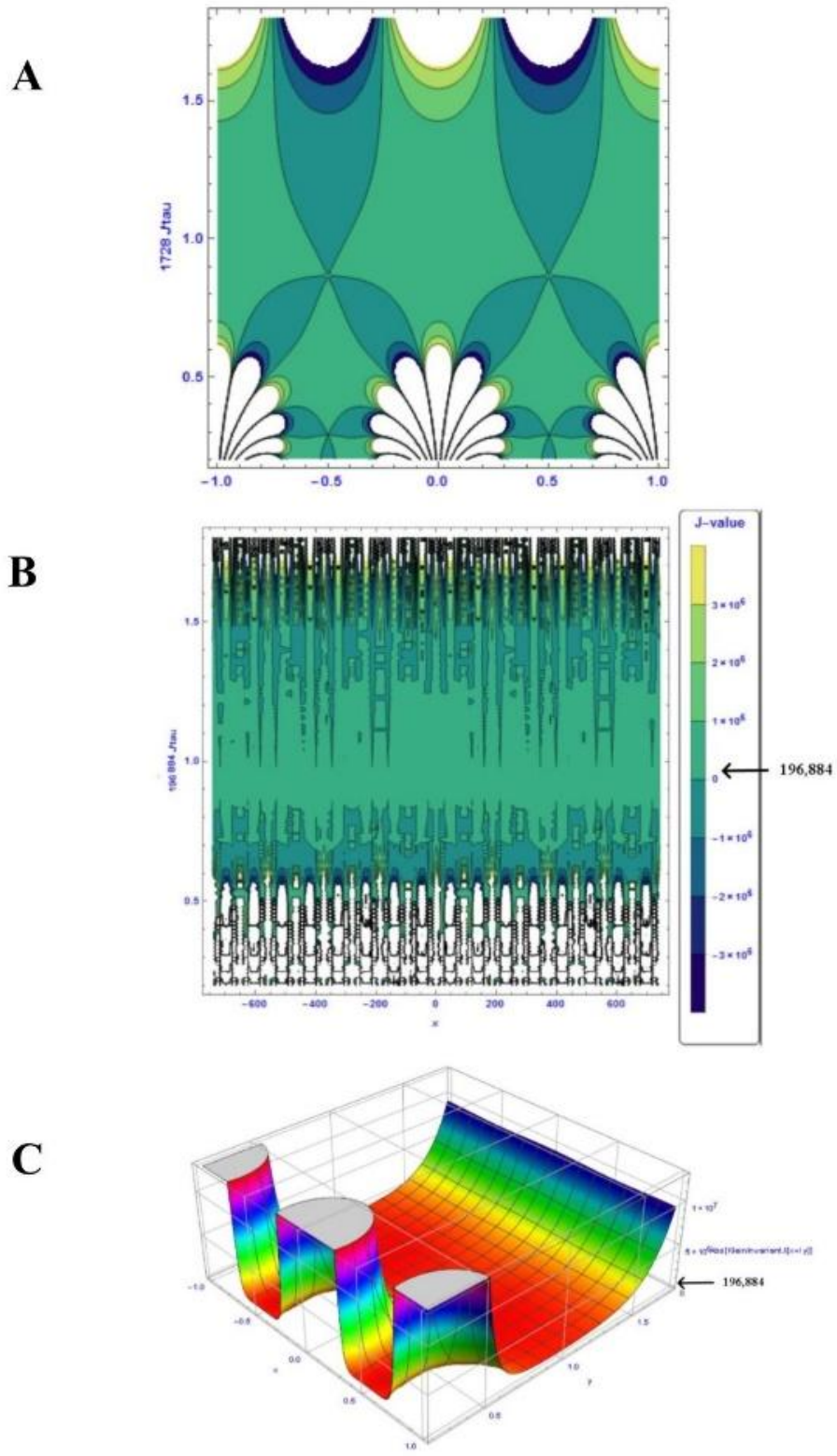

Figure 1. Plots of Klein j-functions on the upper half plane of the complex numbers. **Figure 1A** and **1B** illustrate the 2D plots achieved for j=1728 and 196,884, respectively. Note the different coarse-grained appearance of the two plots. In **Figure 1B**, the olive green area corresponds to the plane where j=196,844. **Figure 1C** displays the 3D plot, in case j=196,844.

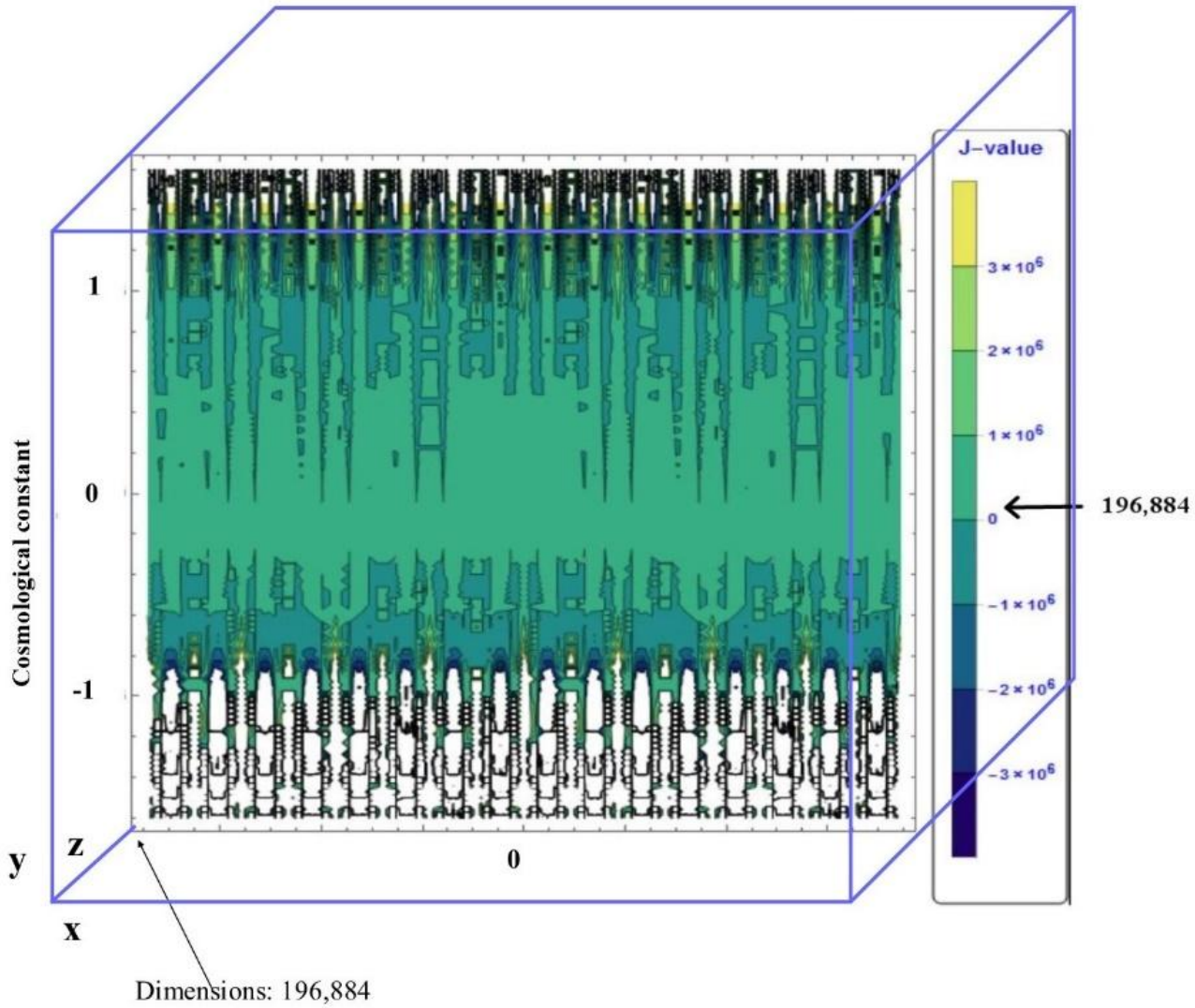

Figure 2. The Monster Module embedded in the upper half plane of the complex numbers. The figure displays the possible physical counterparts. The value 0 on the y-axis is the zone in which our current Universe lie. See text for further details.

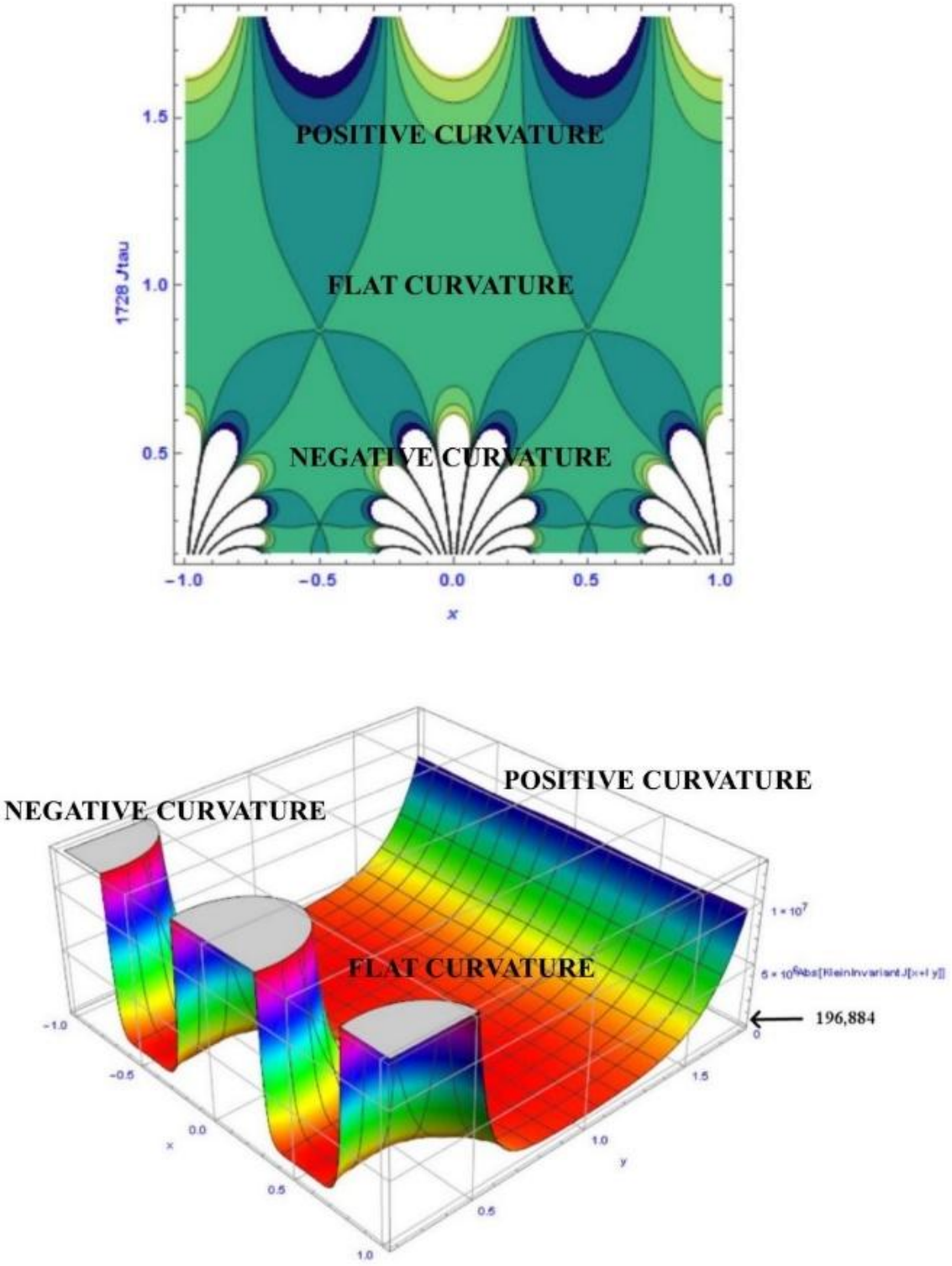

Figure 3. Curvatures in different zones of the j-function plot. The hyperbolic, flat and convex curvatures displayed by j-functions allow us to assess a physical counterpart, e.g., the spatial curvature index, of the abstract meromorphic functions on the upper half plane of the complex numbers.